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CANTOR–BENDIXSON DEGREES AND CONVEXITY IN $\mathbb{R}^2$

BY

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ABSTRACT

We present an ordinal rank, δ^3, which refines the standard classification of non-convexity among closed planar sets. The class of closed planar sets falls into a hierarchy of order type $\omega_1 + 1$ when ordered by δ-rank.

The rank $\delta^3(S)$ of a set S is defined by means of topological complexity of 3-cliques in the set. A 3-clique in a set S is a subset of S all of whose unordered 3-tuples fail to have their convex hull in S. Similarly, $\delta^n(S)$ is defined for all $n > 1$.

The classification cannot be done using δ^2, which considers only 2-cliques (known in the literature also as "visually independent subsets"), and in dimension 3 or higher the analogous classification is not valid.

1. Introduction

Let S be a set in a linear space and suppose that S is not convex. One would like to measure how far S is from being convex. The most natural number for measuring non-convexity of a set S is the least number of convex subsets of S needed to cover S. Let us, then, define $\gamma(S)$ as the least cardinality of a collection of convex sets whose union equals S. The function γ is adopted as the basic measurement of non-convexity. Classification by γ gives countably many different classes of sets with finite γ and (potentially) only two classes with infinite γ: sets with countable γ and sets with uncountable γ.

In this paper we define for each $n > 1$ a degree functions δ^n, and show that δ^3 refines the γ-classification for closed, planar sets. The class

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$\{S: S \subseteq \mathbb{R}^2$ is closed and $\gamma(S) \leq \aleph_0\}$ is divided by δ^3 into $\aleph_1$ sub-classes, while $\{S: S \subseteq \mathbb{R}^2$ is closed and $\gamma(S) > \aleph_0\}$ is a single δ^3-class.

The first step in understanding the structure of a set S with $\gamma(S) = \lambda$ is to understand why S fails to decompose into a union of fewer than λ convex sets.

There is an easy *sufficient* condition for S *not* to be a union of fewer than λ convex sets: the existence of a subset $P \subseteq S$ of cardinality λ, with the property that for any two points in P the line segment connecting them is not contained in S. No two of those points can sit in the same convex subset of S, hence S is not a union of n convex sets. Call a subset of S with this property "visually independent". Let $\alpha(S)$ be the supremum of cardinalities of all visually independent subsets in S.

Does α measure non-convexity adequately? This can be rephrased as whether there exists a "reasonable" function f so that $\gamma(S) \leq f(\alpha(S))$.

For general sets this is badly false (see [6], Section 5), and also in "nice" sets in dimension 3 or higher the connection between α and γ is not well behaved. Nevertheless, closed sets in $\mathbb{R}^2$ show some tight connections between α and γ. A long sequence of results [4, 9, 1, 2] culminated in the discovery [3] that $\gamma(S) \leq f(\alpha(S))$ for some function f, for closed planar sets. Later it was shown that f is at most n^6 in [8]. Recently, n^6 was lowered to $18n^3$ by Matousek and Valtr in [7], where also a lower bound of $O(n^2)$ was set.

In sets which are not a finite union of convex sets, the connection between α and γ is not as tight. There exist compact, planar sets with countable α and uncountable γ ([6], Example 2.1). Put differently, the class of closed planar sets with countable α contains also sets with uncountable γ. This means that the notion of visual independence does not capture all the information as to why a closed $S \subseteq R^2$ cannot be covered by countably many convex subsets.

However, a generalization of visual independence does. Call a subset $P \subseteq S$ a **3-clique** if every 3-element subset $X \subseteq P$ satisfies that its convex closure is not contained in S. Theorem 2.2 in [6] says that a closed set in the plane is not a countable union of convex sets if, and only if, it contains an uncountable (actually perfect) 3-clique. Namely, the *only* reason for such S not to be a countable union of convex sets is that it contains an uncountable 3-clique.

Since, by this theorem, the full information about non-convexity of a closed planar set is stored in the collection of its 3-cliques, it is natural to try and classify non-convexity of such sets by classifying their 3-cliques. It turns out that the standard topological classification of closed countable sets — the Cantor–Bendixson degree — indeed works: for every closed planar set which is a countable union of convex sets there exists a countable ordinal which bounds the Cantor–Bendixson degrees of all 3-cliques in S. This ordinal is, then, the degree of non-convexity of S.

1.1 STATEMENT OF THE RESULTS. Let S be a subset of a linear space. Call $P \subseteq S$ an **n-clique in S** if $n \geq 2$ and for every n-element subset X of P the convex closure of X is not contained in S. Let $\delta^n(S)$ be the supremum of Cantor–Bendixson degrees of all n-cliques in S. Since every n-clique is also an $n+1$-clique, $\delta^n(S) \leq \delta^{n+1}(S)$ for all $n \geq 2$. The rank $\delta(S)$ is the supremum over Cantor–Bendixson degrees of n-cliques for all n, that is, $\delta(S) = \sup\{\delta^n(S) : n < \omega\}$.

It is proved that for every closed set in a polish linear space, an uncountable δ implies an uncountable γ. Surely, if $\delta(S)$ is uncountable then $\delta^n(S)$ is uncountable for some n. In the case of closed planar sets this n has to be ≤ 3, by [6], theorem 2.2. Thus (Corollary 6 below), for a closed planar S, the rank $\delta^3(S)$ is countable if and only if S is a countable union of convex sets. In other words, δ^3 refines γ on closed planar sets by breaking the class of closed $S \subseteq \mathbb{R}^2$ to $\aleph_1 + 1$ classes, so that the top class is that of sets with uncountable γ and all smaller classes stratify the class of sets with countable γ. Since for countable sets the rank δ^n clearly coincides with the usual Cantor–Bendixson degree, any closed set of Cantor–Bendixson degree α is an example of a set with $\delta^3(S) = \alpha$; is is easy to construct uncountable sets of degree α as well.

Perles' Example 2.1 in [6], in which γ is uncountable yet δ^2 equals 1, shows that one cannot get similar classification with δ^2 instead of δ^3. In dimension $d > 2$ a compact set may have δ-rank 1 but still have an uncountable γ (see [6], Example 4.2). Thus, Corollary 6 is sharp in two senses: first, δ^3 cannot be weakened to δ^2 and $\mathbb{R}^2$ cannot be replaced by $\mathbb{R}^3$.

The last remark suggests that classification of closed, non-convex sets in $\mathbb{R}^3$ requires other methods. A more complicated rank function is needed to classify non-convexity of closed sets in *all* Polish vector spaces. Such rank function exists and will be presented in [5].

1.2 HISTORY. Infinite unions of convex sets were studied in [6]. We refer the reader to that paper for basic facts and examples concerning such sets.

The following problem is still open:

PROBLEM 1: *Is it true that a closed planar set in which the closure of every visually independent subset is countable, is a countable union of convex sets?*

This problem was asked by G. Kalai and only a very minor step towards solving it has been ([6], Theorem 4.2).

1.3 Notation: Our notation is standard, except maybe for denoting the set of natural numbers by ω. A topological space X is **polish** if it is complete, metric and separable. A **sequence** is a function whose domain is an initial segment of ω. By n^ω we denote the space of all infinite sequences over n symbols and by n^k the set of sequences of length k over n symbols. The space n^ω is topologized by declaring the set of all infinite sequences that extend a given finite sequence as a basic open set. This topology is polish, by the metric which assigns to two sequences $\eta, \nu \in n^\omega$ the distance $1/k$ where k is the first coordinate in which η and ν are different. We write $\eta \lhd \nu$ to denote that the sequence η is an initial segment of the sequence ν, and by $\eta\widehat{\ }\nu$ the concatenation of η with ν is denoted.

2. Cantor–Bendixson degrees and convexity

We begin by recalling the definition of the Cantor–Bendixson degree of a set S in some topological space X. A point $x \in S$ is isolated in S, if there is an open neighborhood $u \ni x$ so that $S \cap u = \{x\}$. By induction on ordinals define the α-th derived subset of S:

1. $S^{(0)} = S$,
2. $S^{(\alpha+1)} = S^{(\alpha)} - \{x : x \text{ is isolated in } S^{(\alpha)}\}$,
3. If α is limit, then $S^{(\alpha)} = \bigcap_{\beta<\alpha} S^{(\beta)}$.

Let $\mathrm{rk}(S)$, the **Cantor–Bendixson degree** of S, be the least ordinal α for which $S^{(\alpha)} = S^{(\alpha+1)}$. Thus, for example, the Cantor-Bendixson degree of a set which is dense in itself is 0.

FACT 2: If S is a subset of a polish space and $\mathrm{rk}(S) = \alpha + 1$, then there is a closed subset $C \subseteq S$ with $\mathrm{rk}(C) = \mathrm{rk}(S)$.

Given a set S and a point $x \in S$, the **degree of x in S**, which we denote by $\mathrm{rk}_S(x)$, is the last ordinal α for which $x \in S^{(\alpha)}$, if x does not belong to $S^{(\alpha)}$ for all α; if $x \in S^{(\alpha)}$ for all α, we say that $\mathrm{rk}_S(x) = \infty$. Clearly, for every set S and $\beta < \mathrm{rk}(S)$ there are points $x \in S$ with $\mathrm{rk}_S(x) = \beta$ (but $S^{\mathrm{rk}(S)}$ may be null).

We remark that a separable metric space is second countable, and therefore the Cantor–Bendixson degree of every set in such a space is always *countable*.

3. Proofs

Definition 3: Let S be a set in a topological vector space. Let $\delta^n(S)$ be the supremum over all Cantor-Bendixson degrees of n-cliques in S. Let $\delta(S) := \sup\{\delta^n(S) : n < \omega\}$.

THEOREM 4: *Suppose that S is a closed set in a polish linear space E and $\gamma(S) \leq \aleph_0$. Then $\delta(S) < \omega_1$.*

Proof: Suppose that $S \subseteq E$, E is a polish linear space and $\delta(S) = \omega_1$. Let $n \geq 2$ be the least so that $\delta^n(S) = \omega_1$. We may assume, then, that there are closed n-cliques of unbounded (countable) Cantor–Bendixson degrees in S.

LEMMA 5: *Suppose that u is an open neighborhood in E and that u contains n-cliques in S of unbounded Cantor–Bendixson degrees. Then there exist open neighborhoods $u_0, \ldots, u_{n-1}$ such that for every $i < n$, $\mathrm{cl}\, u_i \subseteq u$, u_i contains cliques of unbounded Cantor–Bendixson degrees and so that for every choice of $y_i \in \mathrm{cl}\, u_i$, $\mathrm{conv}(y_0, \ldots, y_{n-1}) \not\subseteq S$.*

Proof of Lemma 5: Fix a countable base $\mathcal{B}$ for the topology of E (e.g. all balls of rational radius and a center in some countable dense set).

Define now a mapping from ω_1 to n-tuples from $\mathcal{B}$, $\beta \mapsto (u_i^\beta, \ldots, u_{n-1}^\beta)$, as follows. Let $\beta < \omega_1$ be given. Choose first an n-clique $P \subseteq u$ and n points in P, $x_0^\beta, \ldots, x_{n-1}^\beta$, such that $\mathrm{rk}_P(x_i^\beta) \geq \beta$. Since the complement of S is open, there are open neighborhood u_i of x_i^β for $i < n$ so that for every choice of $y_i \in u_i$ it holds that $\mathrm{conv}(y_0, \ldots, y_{n-1}) \not\subseteq S$. By shrinking each u_i, we may assume that $u_i \in \mathcal{B}$, $\mathrm{cl}\, u_i \subseteq u$ and that $\mathrm{conv}(y_0, \ldots, y_{n-1}) \not\subseteq S$ for every choice of $y_i \in \mathrm{cl}\, u_i^\beta$. Let $(u_i^\beta, \ldots, u_{n-1}^\beta) := (u_0, \ldots, u_{n-1})$.

Since there are only countably many n-tuples from $\mathcal{B}$, there is a fixed n-tuple $(u_0, \ldots, u_{n-1})$ and an unbounded $I \subseteq \omega_1$ so that $(u_0, \ldots, u_{n-1}) = (u_0^\beta, \ldots, u_{n-1}^\beta)$ for every $\beta \in I$.

Therefore, for every $i < n$ and an ordinal $\beta < \omega_1$, there exists a closed clique P and a point $x \in P \cap u_i$ with $\mathrm{rk}_P(x) \geq \beta$. Since u_i is open, $\mathrm{rk}(P \cap u_i) \geq \beta$. Therefore each u_i contains cliques of unbounded degrees. ∎

Suppose that S is closed, $\gamma(S) \leq \aleph_0$ and S contains n-cliques of unbounded degrees. By induction on k define neighborhoods u_η for $\eta \in n^k$ so that:

1. $d(u_\eta) < 1/k$ for all $\eta, \nu \in n^k$,
2. $\eta \lhd \nu \Rightarrow \mathrm{cl}\, u_\nu \subseteq u_\eta$,
3. u_η contains closed cliques in S of unbounded Cantor–Bendixson degrees,
4. if $\eta_0, \ldots, \eta_{n-1}$ are distinct and agree up to $k-1$, then for every choice of y_i from $\mathrm{cl}\, f(\eta_i)$ the convex closure of $\{y_0, \ldots, y_{n-1}\}$ is not contained in S.

At stage $k+1$ use Lemma 5 to find, for each $\eta \in n^k$, sub-neighborhoods $\{u_{\eta^\frown i} : i < \kappa\}$, of u_η, which satisfy conditions 1–4 above.

Suppose now that u_η is defined for every finite sequence over n and define $g : n^\omega \to S$ by $g(\eta) := \bigcap_k u_{\eta\restriction k}$. Since E is complete, g is well defined. Since S is closed, $g(\eta) \in S$ for every $\eta \in n^\omega$.

Suppose now that $S = \bigcup_n C_n$. The space n^ω of all infinite sequences over n symbols is a complete separable metric space under the metrics $d(\eta, \nu) = 1/k$ for the least k such that $\eta(k) \neq \nu(k)$. By the Baire category theorem, there is some index m so that $f^{-1}(C_m)$ is somewhere dense. Choose some k and a sequence $\nu \in n^k$ so that $f^{-1}(C_m)$ is dense in $\{\eta \in n^\omega : \nu \lhd \eta\}$. For every $i < n$ there must then be a sequence η_i so that $\eta_i \restriction k+1 = \eta\frown i$, and $f(\eta_i) \in C_m$. But then $f(\eta_i) \in u_{\eta\frown i}$ by the definition of g, and therefore $\mathrm{conv}(g(\eta_0), \dots, g(\eta_{n-1})) \not\subseteq S$ by condition 4. Therefore C_m is not convex. ∎

COROLLARY 6: *A closed planar set is a countable union of convex sets if and only if* $\delta^3(S) < \omega_1$.

Proof: One direction is proved above.

For the other direction, suppose that $S \subseteq \mathbb{R}^3$ is closed and is not a countable union of convex sets. By [6], Theorem 2.2 there is a perfect 3-clique $P \subseteq S$. Every subset of P is a 3-clique, and since P is perfect it contains countable sets of unbounded Cantor–Bendixson degrees. ∎

We observe that Example 2.1 in [6] of a compact planar set S satisfies that $\delta^2(S) = 1$ while $\delta^3(S) = \omega_1$. Hence, classification by δ^2 does not refine the classification by γ.

It is natural to ask at this point whether $\delta^4(S)$ classifies non-convexity of closed sets in $\mathbb{R}^3$ analogously to the manner δ^3 classifies closed sets in $\mathbb{R}^2$. This is false by Example 4.1 in [6] (see also [5]). This example is of a compact $S \subseteq \mathbb{R}^3$ with $\delta(S) = 1$ and $\gamma(S) > \aleph_0$.

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